Math 10A with Professor Stankova
Worksheet, Discussion #12; Friday, 9/22/2017
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Taylor Polynomials

Examples

1. Use the second order Taylor series to approximate $\sqrt{17}$.

Solution: The formula for the second order Taylor series expanded at $x = c$ is

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2.$$

The closest square is 16, so we can expand around there since we know $\sqrt{16} = 4$. We have that $f(x) = \sqrt{x}, f'(x) = \frac{1}{2\sqrt{x}}, f''(x) = \frac{-1}{4x\sqrt{x}}$. Plugging in $c = 16$, we have that

$$f(x) \approx 4 + \frac{x-16}{8} - \frac{1}{512}(x-16)^2.$$

Plugging in $x = 17$, we have that

$$\sqrt{17} \approx 4 + \frac{1}{8} - \frac{1}{512} \approx 4.123.$$

2. Find the Taylor series for $x^5 + 3x^3 + 2x + 10$.

Solution: Taylor series give you a polynomial approximation for your function. But if your function is already a polynomial, then it gives the same thing. Try it out and verify it by yourself! So the Taylor series is just $x^5 + 3x^3 + 2x + 10$.

Problems

3. Use the second order approximation to $\sqrt[3]{28}$.

Solution: A close cube that we know is $3^3 = 27$. So we calculate the second order Taylor series expanded at $x = 27$ to get

$$\sqrt[3]{x} \approx 3 + \frac{x-27}{27} - \frac{(x-27)^2}{2187}.$$

So plugging in 28 gives

$$\sqrt[3]{28} \approx 3 + \frac{1}{27} - \frac{1}{2187} \approx 3.036.$$

4. Use the second order approximation to find $\ln 1.1$.

Solution: We know that $\ln 1 = 1$. So we can expand out at $x = 1$ to get

$$\ln x \approx 0 + (x-1) - \frac{(x-1)^2}{2}.$$

Thus, we have that $\ln 1.1 \approx (0.1) - \frac{0.1^2}{2} = 0.095$.

5. Use the second order approximation to find $\sqrt{5}$.

Solution: We have that $\sqrt{4} = 2$ and 4 is close to 5 so we expand there. We have that

$$\sqrt{x} \approx 2 + \frac{x-4}{4} - \frac{1}{64}(x-4)^2.$$

Now we plug in 5 to get

$$\sqrt{5} \approx 2 + \frac{1}{4} - \frac{1}{64} \approx 2.234.$$

6. Use the second order approximation to find $e^{0.1}$.

Solution: We know that $e^0 = 1$ so we can expand around $x = 0$. Doing so gives

$$e^x \approx 1 + x + \frac{x^2}{2}.$$

Thus, we have that $e^{0.1} \approx 1 + 0.1 + 0.1^2/2 = 1.105$.

7. Use the second order approximation to find $\sec(0.1)$.

Solution: We know that $\sec(0) = 1/\cos(0) = 1$. We can expand there using the fact that the first derivative is $\sec(x)\tan(x)$ and the second derivative is $\sec(x)(\tan^2(x) + \sec^2(x))$. Thus, we get that the Taylor series is

$$\sec(x) \approx 1 + \frac{x^2}{2}.$$

Thus, we have that $\sec(0.1) \approx 1 + 0.1^2/2 = 1.005$.

8. Use the third order approximation to find $\sin(0.1)$.

Solution: We expand around 0 since $\sin 0 = 0$. We find that

$$\sin x \approx x - \frac{x^3}{6},$$

and so $\sin(0.1) \approx 0.1 - 0.1^3/6 = 0.0998$.

9. Use the second order approximation to find $\cos(0.1)$.

Solution: Expanding at $x = 0$ gives

$$\cos(x) \approx 1 - \frac{x^2}{2}.$$

Thus, $\cos(0.1) \approx 1 - 0.1^2/2 = 0.995$.

Newton's Method

Examples

10. Find the roots of $f(x) = x^3 - x + 1$.

Solution: Taking the derivative, we get that the derivative is $3x^2 - 1$. This has roots at $\pm 1/\sqrt{3}$. When we plug in $1/\sqrt{3}$, we get that $f(1/\sqrt{3}) = 1 - 2/3\sqrt{3} > 0$. Thus, this function only has one zero because the local minimum at $x = 1/\sqrt{3}$ is positive.

Since $x = -1/\sqrt{3}$ is a local maximum, we know that the zero must be $< -1/\sqrt{3}$. We can start by guessing $x = -2$. The formula for Newton's method gives us

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} = x - \frac{3x^2 - 1}{x^3 - x + 1}.$$

Plugging in $x = -2$ gives us $\frac{-17}{11}$. So the root is approximately -1.5454, the real root is -1.32.

Problems

11. Use Newton's method to estimate $\sqrt[4]{16.32}$.

Solution: This value is a root of $x^4 - 16.32 = 0$. We can start at $x = 2$ and using Newton's method gives us

$$x' = x - \frac{f(x)}{f'(x)} = 2 - \frac{-.32}{4 \cdot 2^3} = 2 + 0.01 = 2.01.$$

The real answer is about 2.0099.

12. Find the critical points of $g(x) = \sin(x) - x^2$

Solution: We want to find when the derivative is 0 or when $f(x) = \cos(x) - 2x = 0$. Taking the derivative again, we find that it is $-\sin(x) - 2 < 0$ for all x. So this function is always decreasing and has a unique root. We plug in $x = 0$ to start, then calculate

$$x' = x - \frac{f(x)}{f'(x)} = -\frac{1}{-2} = \frac{1}{2}.$$

The real solution is ≈ 0.45.

13. Find the critical points of $e^x + x^2$.

Solution: The critical points are when the derivative is 0 so $f(x) = e^x + 2x = 0$. Taking the second derivative, we have that $f'(x) = e^x + 2 > 0$ so this is an always increasing function. Therefore, it will only have 1 zero. We plug in the only value that we know of $x = 0$ and get

$$x' = x - \frac{f(x)}{f'(x)} = 0 - \frac{1}{3} = -\frac{1}{3}.$$

The real solution is ≈ -0.35.

14. Find when $\cos x = x$.

Solution: Notice that when taking the derivative of $\cos x - x$, we get $\sin x - 1 \leq 0$ so this is a decreasing function which has at most on zero. We start at $x = 0$ to get the next point

$$x' = x - \frac{f(x)}{f'(x)} = 0 - \frac{1}{-1} = 1.$$

The real solution is ≈ 0.739.

15. Use Newton's method to estimate $\sqrt[3]{28}$.

Solution: We want to find the root of $x^3 - 28$. We guess $x = 3$ and get that the next point is

$$x' = x - \frac{f(x)}{f'(x)} = 3 - \frac{-1}{27} = \frac{82}{27} \approx 3.037.$$

The real solution is ≈ 3.0366.

16. Use Newton's method with two steps to estimate $\sqrt{5}$.

Solution: We want to find the root of $x^2 - 5 = 0$. The first guess is $x = 2$ and the next point is

$$x' = 2 - \frac{-1}{4} = \frac{9}{4}.$$

Doing that again, we get that the next point is

$$x' = \frac{9}{4} - \frac{81/16 - 5}{9/2} \approx 2.2361.$$

The real answer is approximately 2.23607.

17. Use Newton's method to estimate $2^{0.1}$.

Solution: We can rewrite this as $2^{1/10}$ so we want to find a root of $x^1 0 - 2 = 0$. Using Newton's method with a guess of 1 gives us

$$x' = 1 - \frac{-1}{10} = 1.1.$$

The real answer is ≈ 1.0718.

L'Hopital's Rule

Examples

18. Find $\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{3x}$.

Solution: We use the trick of turning exponents into products by taking e to the ln of the function. So doing this gives

$$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{3x} = \lim_{x\to\infty}\exp\left[\ln\left(1+\frac{1}{2x}\right)^{3x}\right] = \exp\left[\lim_{x\to\infty}3x\ln\left(1+\frac{1}{2x}\right)\right].$$

Plugging in ∞ gives $\infty\cdot 0$ which is a product indeterminate and so we can turn this product into a quotient. Doing so gives

$$\lim_{x\to\infty}3x\ln\left(1+\frac{1}{2x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{2x}\right)}{(3x)^{-1}} = \lim_{x\to\infty}\frac{\frac{1}{1+1/2x}(-2(2x)^{-2})}{-3(3x)^{-2}}$$
$$= \lim_{x\to\infty}\frac{3}{2+\frac{1}{x}} = \frac{3}{2}.$$

Thus the answer to the original limit is $e^{3/2}$.

19. Find $\lim_{x\to\infty}(x^2-\ln\sqrt{x})$.

Solution: Plugging in ∞ gives $\infty-\infty$ which is indeterminate. We can't use L'Hopitals rule just yet as we need to express the answer as a quotient. We can write

$$x^2-\ln\sqrt{x} = \frac{1}{x^{-2}}-\ln\sqrt{x} = \frac{1-x^{-2}\ln\sqrt{x}}{x^{-2}}.$$

Thus in order to calculate the original limit, we need to calculate the limit of $x^{-2}\ln\sqrt{x} = \frac{\ln\sqrt{x}}{x^2}$. This is indeterminate by L'Hopital's rule and so we can calculate the derivative as

$$\lim_{x\to\infty}\frac{\ln\sqrt{x}}{x^2} = \lim_{x\to\infty}\frac{\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}}{2x} = \lim_{x\to\infty}\frac{1}{4x^2} = 0.$$

Plugging this back into our original equation, we have that

$$\lim_{x\to\infty}x^2-\ln\sqrt{x} = \lim_{x\to\infty}\frac{1-x^{-2}\ln\sqrt{x}}{x^{-2}} = \frac{1-0}{0} = \infty.$$

Problems

20. Find $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$.

Solution: Plugging in $x = 4$ gives $0/0$ which is indeterminate. Now we use LHopitals rule to get

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \frac{1}{1/(2\sqrt{x})} = \lim_{x\to 4} 2\sqrt{x} = 4.$$

21. Find $\lim_{x\to 0} \frac{3^x - 2^x}{x^2 - x}$.

Solution: Plugging in 0 gives $0/0$ which is indeterminate, so we can use Lhopitals. This gives

$$\lim_{x\to 0} \frac{3^x - 2^x}{x^2 - x} = \lim_{x\to 0} \frac{e^{x\ln 3} - e^{x\ln 2}}{x^2 - x} = \lim_{x\to 0} \frac{\ln 3 \cdot e^{x\ln 3} - \ln 2 \cdot e^{x\ln 2}}{2x - 1}$$

$$= \frac{\ln 3 - \ln 2}{-1} = \ln 2 - \ln 3.$$

22. Find $\lim_{x\to 0} \frac{x\tan x}{\sin 3x}$.

Solution: Plugging in $x = 0$ gives $0/0$ so using Lhopitals gives

$$\lim_{x\to 0} \frac{x\tan x}{\sin 3x} = \lim_{x\to 0} \frac{x\sec^2(x) + \tan x}{3\cos(3x)} = \frac{0}{3} = 0.$$

23. Find $\lim_{x\to 0} \frac{\sin(x^2)}{x\tan x}$.

Solution: Plugging in 0 gives $0/0$ and so we can use LHopitals rule to get

$$\lim_{x\to 0} \frac{\sin(x^2)}{x\tan x} = \lim_{x\to 0} \frac{2x\cos(x^2)}{x\sec^2(x) + \tan x}.$$

Plugging in 0 again gives $0/0$ yet again, so we use LHopital's again to get

$$\lim_{x\to 0} \frac{2x\cos(x^2)}{x\sec^2(x) + \tan x} = \lim_{x\to 0} \frac{2\cos x^2 - 4x^2\sin x^2}{2\sec^2(x) + 2x\tan x\sec^2(x)} = \frac{2-0}{2+0} = 1.$$

24. Find $\lim_{x\to 0} \frac{x^2 e^x}{\tan^2 x}$.

Solution: Plugging in 0 gives 0/0 so we use LHopitals to get

$$\lim_{x\to 0} \frac{x^2 e^x}{\tan^2 x} = \lim_{x\to 0} \frac{2xe^x + x^2 e^x}{2\tan x \sec^2 x}.$$

Plugging in 0 again gives 0/0 so we use LHopitals again to get

$$\lim_{x\to 0} \frac{2xe^x + x^2 e^x}{2\tan x \sec^2 x} = \lim_{x\to 0} \frac{x^2 e^x + 4xe^x + 2e^x}{2\tan x(2\sec x \cdot \sec x \tan x) + 2\sec^4 x} = \frac{2}{2} = 1.$$

25. Find $\lim_{x\to\infty} (\sqrt{x^2+1} - \sqrt{x+1})$.

Solution: Plugging in ∞ gives $\infty - \infty$ which is an indeterminate. This is not a quotient so we can't use LHopital's yet. But we can try to multiply by the conjugate to get

$$\lim_{x\to\infty} (\sqrt{x^2+1} - \sqrt{x+1}) = \lim_{x\to\infty} \frac{x^2+1-(x+1)}{\sqrt{x^2+1}+\sqrt{x+1}}.$$

Now we plug in ∞ to get ∞/∞ so we can use LHopitals and get

$$= \lim_{x\to\infty} \frac{2x-1}{x/\sqrt{x^2+1} + 1/(2\sqrt{x+1})} = \lim_{x\to\infty} \frac{2x-1}{1/\sqrt{1+1/x^2} + 1/(2\sqrt{x+1})} = \frac{\infty}{1+0} = \infty.$$

Note that we could have solved it after multiplying by the conjugate by dividing the top and bottom by the largest power of x we saw, which was x^2. Doign so gives

$$\text{dim}_{x\to\infty} \frac{x^2 - x}{\sqrt{x^2+1}+\sqrt{x+1}} = \lim_{x\to\infty} \frac{1-1/x}{\sqrt{1/x^2+1/x^4}+\sqrt{1/x^3+1/x^4}} = \infty/0 = \infty.$$

26. Find $\lim_{x\to 0^+} \ln x \cdot \tan x$.

Solution: Plugging in 0 gives $(-\infty)\cdot 0$. So, we can write it as

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+} \frac{1/x}{-\csc^2(x)} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x}.$$

Plugging in 0 gives 0/0 so we can use LHopitals again to get

$$= \lim_{x\to 0^+} \frac{-2\sin x \cos x}{1} = 0.$$

27. Find $\lim_{x\to 0^+} x^{\sin x}$.

Solution: We don't like having x raised to some function of x so we do our trick of taking e to the ln of the function. This gives

$$\lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} \exp(\ln x^{\sin x}) = \exp\left[\lim_{x\to 0^+} \sin x \ln x\right].$$

Calculating the inner limit gives

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\csc(x)} = \lim_{x\to 0^+} \frac{1/x}{-\cot(x)\csc(x)} = \lim_{x\to 0^+} \frac{-\sin^2(x)}{\cos(x)x}.$$

Plugging in 0 again gives 0/0 so we use LHopitals again to get

$$\lim_{x\to 0^+} \frac{-2\sin x \cos x}{\cos x - x \sin x} = 0.$$

So our original answer is $e^0 = 1$.